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Carathéodory number and exchange number in Δ -convexity. (English) Zbl 1581.05135
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A (finite) convexity space is a finite set X equipped with a collection $\mathcal{C}$ of subsets of X such that $\emptyset \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The members of $\mathcal{C}$ are said to be convex sets. Given a subset S of X , its convex hull, $\langle S \rangle$, is defined to be the intersection of all convex sets containing S . This article considers the notion of Δ -convexity in graphs: given a (finite, simple, undirected) graph G , a subset S of $V(G)$ is Δ -convex if, whenever $v \in V(G)$ forms a triangle with two vertices in S , we have $v \in S$. The collection of Δ -convex sets in G forms a convexity space, which we shall call a Δ -convexity space on G .

Given a convexity space $(X, \mathcal{C})$, several parameters measure its “dimension”. A subset S of X is said to be Carathéodory independent if $\langle S \rangle \setminus \bigcup_{a \in S} \langle S \setminus \{a\} \rangle \neq \emptyset$. A subset S of X is said to be exchange independent if S is a singleton or if there exists $p \in S$ such that $\langle S \setminus \{p\} \rangle \setminus \bigcup_{a \in S \setminus \{p\}} \langle S \setminus \{a\} \rangle \neq \emptyset$. The Carathéodory number $c_\Delta(G)$ (resp., exchange number $e_\Delta(G)$) of a convexity space is the maximum size of a Carathéodory independent (resp., exchange independent) set in X . In this article, the authors compute the Carathéodory and exchange numbers for Δ -convexity spaces on chordal graphs, block graphs, and graphs obtained as a box product, $G \square H$, strong product, $G \boxtimes H$, or lexicographic product, $G \circ H$.

The authors prove that if G is a graph with k triangles, then $c_\Delta(G) \leq k+1$. This bound is tight in general, and moreover, for every $n \in \mathbb{N}$ there is a graph G with $c_\Delta(G) = n$. If G is a block graph whose blocks all lie on a single chain, and no block is isomorphic to K_2 , then $c_\Delta(G) = \ell + 1$, where ℓ is the number of blocks of G . The authors also prove that if G is a 2-connected chordal graph, then $c_\Delta(G) = 2$. Next, they show that if G and H are graphs such that $c_\Delta(G), c_\Delta(H) > 2$, then $c_\Delta(G \square H) \geq c_\Delta(G)c_\Delta(H)$, and this lower bound is tight in general. If G and H are connected graphs on at least two vertices, then $c_\Delta(G \boxtimes H) = 2 = c_\Delta(G \circ H)$. Analogous results are also proved for $e_\Delta(G)$.

In Theorems 2.9 and 3.10, the computations of the Carathéodory and exchange numbers, respectively, for block graphs whose blocks do not all lie on a single chain are incorrect. Lemma 2.6(iii) is not true for all graphs, unless $\{u, v\} \notin E(G)$. The proof of Theorem 2.3, which states that $c_\Delta(G) \leq k + 1$ when G has k triangles, seems to require further elaboration as the claim in the last paragraph that $\langle S' \rangle = \bigcup_{i=1}^j (\langle S' \setminus \{u_i\} \rangle \cup \langle S' \setminus \{v_i\} \rangle)$ does not appear to be entirely straightforward.

Other minor typos are as follows. The notation $\langle S \rangle$ introduced before the statement of Theorem 1.1 for the subgraph of G induced by S conflicts with its usage in the rest of the paper to denote the convex hull of S . In the last line of the proof of Theorem 4.2, the union should be taken over all $(a, b) \in S \setminus \{(g, h_i)\}$. In the statement of Theorem 4.3, read “greater than or equal to two” in place of “greater than two”. The last word in the proof of Theorem 4.5 should be “E-dependent”.

Reviewer: [Brahadeesh Sankarnarayanan \(Jodhpur\)](#)

MSC:

- [05C69](#) Vertex subsets with special properties (dominating sets, independent sets, cliques, etc.)
- [05C76](#) Graph operations (line graphs, products, etc.)
- [05C99](#) Graph theory
- [52A01](#) Axiomatic and generalized convexity

Keywords:

convexity space; Carathéodory number; exchange number; graph products; block graphs

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